International Journal of Mathematics and
Computer Applications Research (IJMCAR)
ISSN(P): 2249-6955; ISSN(E): 2249-8060
Vol. 4, Issue 3, Jun 2014, 99-108

NUMERICAL METHODS FOR PARABOLIC PARITIAL DIFFERENTIAL EQUATIONS

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ABSTRACT

Many partial differential equations cannot be solved by analytical methods in closed form solution. In most of the research work in fields like, applied elasticity, theory of plates and shells, hydrodynamics, quantum mechanics etc., the research problems reduce to partial differential equations. Since analytical solutions are not available, we go in for numerical solution of the partial differential equations by various methods.

KEYWORDS: Numerical Analysis, Partial Differential Equations, Analytical Methods

1.1 INTRODUCTION

The general linear partial differential equation of the second order in two independent variables is of the form:

$$A\,(x,y)\frac{\partial^2 u}{dx^2} + B\,(x,y) + \frac{\partial^2 u}{\partial x \partial y} + C(x,y)\,\frac{\partial^2 u}{dy^2} + F\,\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right) = 0$$

Such partial equation is said to be:

(1)Elliptic if $B^2 - 4\,AC < 0$, (2) Parabolic if $B^2 - 4\,AC = 0$,

(3)Hyperbolic if $B^2 - 4\,AC > 0$,

The one dimensional heat equation, namely,

$\frac{\partial u}{\partial t} = \alpha^2\,\frac{\partial^2 u}{\partial x^2}$ Where, $\alpha^2 - \frac{k}{pc}$ is an example of parabolic equation

$\alpha^2 = \frac{1}{a}$, the equation (1.20), becomes, $\frac{\partial^2 u}{\partial x^2} - a\frac{\partial u}{\partial t} = 0,$

Here$A = 1, \quad B = 0, \quad C = 0,$

$\therefore\ B^2 - 4AC = 0$. Therefore, it is parabolic in all points.

The solution of this equation is a temperature function $u\ (x,t)$ which is defined for values of x from 0 to l and for values of time t from 0 to ∞. The solution is not defined in a closed domain but advances in an open-ended region from initial values, satisfying the prescribed boundar

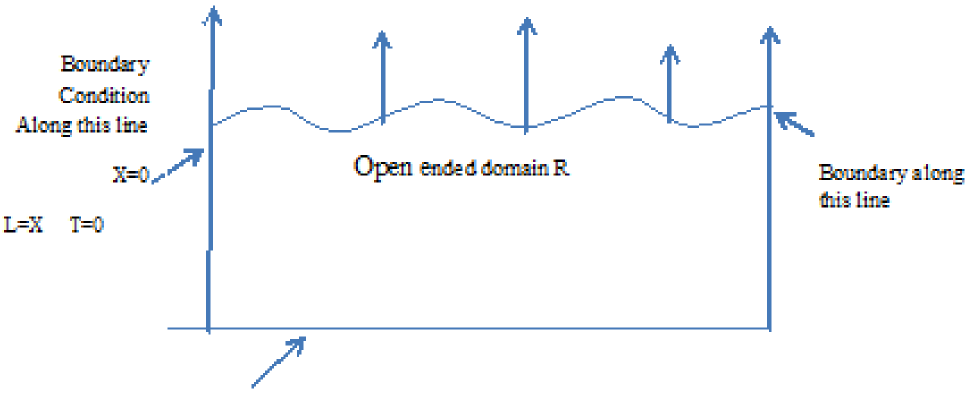

Figure 1: Initial Condition Prescribe

Satisfying the prescribed boundary conditions in general, the study of pressure waves in a fluid propagation of heat and unsteady state problems lead to parabolic type of equations. In this paper we are studying the finite difference algorithm for solving parabolic partial differential equation.

1.2 Two Dimensional Heat Equation

The heat-conduction equation can be applied to more than one spatial dimension. For two dimensions, its form is:

$$\frac{\partial u}{\partial t} = \alpha\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \text{Where, } \alpha = \frac{c^2 k}{h^2}$$

Example: 1 (Solve One Dimensional Heat Equation)

Find the analytical solution of the parabolic equation

$u_{xx} = 2u_t$ When $u(0,t) = u(4,t) = 0$ and $u(x,0) = x(4-x)$, taking $h = 1$ and $c^2 = \frac{1}{2}$,

Find the Values up to $t = 5$

Solution

It is already given parabolic equation $u_{xx} = 2u_t$

$$\frac{1}{2}\frac{\partial^2 u}{\partial x^2} = \frac{du}{dt}$$

and also given the boundary condition, $u(0,t) = u(4,t) = 0$

the initial condition $u(x,0) = x(4-x)$, we can write parabolic equation in the form,

$$\frac{du}{dt} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} \tag{1}$$

Where $c^2 = \frac{1}{2}$

Let $u = XT$ (2)

Where X is the function of x only and T is the function of t only

Differentiating (2) partially $w.r.t.t$, we get

$$\frac{\partial u}{\partial t} = XT' \tag{3}$$

$$\frac{\partial^2 u}{\partial x^2} = X''T \tag{4}$$

Let (3) and (4) equal $-p^2$

$XT' = c^2 X''T = -p^2$

Separating variables, we get

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T'}{T} = -p^2$$

Where $X'' = \frac{d^2X}{dx^2}$ and $T' = \frac{dT}{dt}$

$$T' = -c^2 p^2 T$$

$$\frac{dT}{dt} = -c^2 p^2 T$$

$$DT = -c^2 p^2 T$$

$$m = -c^2 p^2$$

$$T = c_1 e^{-c^2 p^2 t}$$

Now,

$$X'' = -p^2 X$$

$$\frac{d^2 X}{dx^2} = -p^2 X$$

$$D^2 X = -p^2 X$$

$$m = \pm ip$$

$$X = c_2 \cos px + c_3 \sin px$$

We have, $c^2 = \frac{1}{2}$, put the value of X and T in equation (2) we get,

$$u = (c_2 \cos px + c_3 \sin px)\, c_1 e^{\frac{-p^2 t}{2}} \tag{5}$$

Now, put the value of $u(0,t) - 0$, where $x = 0$ in equation (5), we get

$$0 = c_2 c_1 e^{\frac{-p^2 t}{2}}$$

Let $c_2 = 0,$ arbitrary constant we have $T = c_1 e^{-c^2 p^2 t}$ so $c_1 \neq 0,$

$$u = (c_3 \sin px)\, c_1 e^{\frac{-p^2 t}{2}} \tag{6}$$

Put the value of $u(4,t) = 0$, where $x = 4$ in equation (6), we get

$$0 = c_3 \sin 4p\, c_1 e^{\frac{-p^2 t}{2}}$$

$$\sin 4p = 0 = \sin n\pi$$

$$p = \frac{n\pi}{4}$$

Put the value of p in equation (6)

$$u = c_3 \sin(\frac{n\pi}{4})x \;\; c_1 e^{\frac{-(\frac{n\pi}{4})^2 t}{2}}$$

$$u = c_1\, c_3\, sin(\frac{n\pi}{4})x\; e^{\left(\frac{n^2\pi^2}{32}\right)t} \tag{7}$$

Where $c_1\, c_3 = \sum bn$

$$u = \sum b_n\;\; sin(\frac{n\pi}{4})\; x\, e^{-\left(\frac{n^2\pi^2}{32}\right)t} \tag{8}$$

Now we find the value of bn *by Fourier series,*

$$b_n = \frac{2}{l} \int_0^l f(x) \sin(\frac{n\pi x}{l})\, dx$$

$$= \frac{2}{4} \int_0^4 x(4-x) \sin(\frac{n\pi x}{4})\, dx$$

$$b_n = \frac{1}{2} \int_0^4 4x \sin(\frac{n\pi x}{4})\, dx - \frac{1}{2} \int_0^4 x^2\, sin(\frac{n\pi x}{4})\, dx \tag{9}$$

Integration first part from equation (9) by the form of integral,

$$I_1 = 2\left[-x\frac{\cos\left(\frac{n\pi x}{4}\right)}{\left(\frac{n\pi}{4}\right)} + \frac{\sin\left(\frac{n\pi x}{4}\right)}{\left(\frac{n\pi}{4}\right)^2}\right]_0^4$$

$$I_1 = -32\frac{\cos n\pi}{n\pi} \tag{10}$$

Integration second part from equation (9)

$$I_2 = \frac{1}{2}\left[\left\{-x^2\frac{\cos\left(\frac{n\pi x}{4}\right)}{\left(\frac{n\pi}{4}\right)}\right\} - \frac{8}{n\pi}\left\{-x\frac{\sin\left(\frac{n\pi x}{4}\right)}{\left(\frac{n\pi}{4}\right)}\right\} + \frac{8}{n\pi}\left\{\frac{\cos\left(\frac{n\pi x}{4}\right)}{\left(\frac{n\pi}{4}\right)^2}\right\}\right]_0^4$$

$$= \frac{1}{2}\left[-16\frac{\cos n\pi}{\left(\frac{n\pi}{4}\right)} + \frac{8}{n\pi}\left\{\frac{\cos n\pi}{\left(\frac{n\pi}{4}\right)^2}\right\}\right] - \frac{4^3}{n^3\pi^3}[1]$$

$$I_2 = \left[-32\frac{\cos n\pi}{n\pi} + \frac{64}{n^3\pi^3}(\cos n\pi - 1)\right] \tag{11}$$

Putting the value of (10) and (11) in (9) we get,

$$b_n = -32\frac{\cos n\pi}{n\pi} + 32\frac{\cos n\pi}{n\pi} - \frac{64}{n^3\pi^3}(\cos n\pi - 1)$$

$$b_n = -\frac{64}{n^3\pi^3}(\cos n\pi - 1)$$

When $n = 0$

$$b_0 = 0$$

$$b_1 = -\frac{64}{\pi^3}(-2)$$

$$b_1 = \frac{128}{\pi^3}$$

Put value of b_1 in equation (8), we get

$$u = \frac{128}{\pi^3}\sin\left(\frac{n\pi}{4}\right) x\, e^{-\left(\frac{n^2\pi^2}{32}\right)t} \tag{12}$$

Where $n = 1$

$$u_1 = \frac{4.128196}{\sqrt{2}} \times e^{-\left(\frac{\pi^2}{32}\right)t}$$

$$u_1 = \frac{4.128196 \times 0.731615}{\sqrt{2}} = 2.1356$$

$$u_2 = 4.128196 \times e^{-\left(\frac{\pi^2}{32}\right)t}$$

$$u_2 = 3.032585$$

$$u_3 = 4.128196\, sin(\frac{3n\pi}{4})\; e^{-\left(\frac{\pi^2}{32}\right)t}$$

$$u_3 = 2.1356$$

Analytical solution for u_1, u_2, u_3.

NUMERICAL SOLUTION

Now We Solve (Example: 1) by Numerical Methods

Find the solution of the parabolic equation $u_{xx} = 2u_t$ when $u(0,t) - u(4,t) - 0$ and $u(x,0) = x(4-x)$, taking $h = 1$. Find the values up to $t = 5$.

Solution

Given a parabolic equation $u_{xx} = 2u_t$

$$\frac{1}{2}\frac{\partial^2 u}{\partial x^2} = \frac{du}{dt}$$

Also given the boundary condition, $u(0,t) - u(4,t) - 0$

and the initial condition $u(x,0) = x(4-x)$,

when we put the value of $x = 1,2,3,4$ in the relation of the initial condition, it gives that,

$$u(0,0) = 3,\ \ u(1,0) = 4,\ \ \ u(2,0) = 3,\ \ u(3,0) = 3,\ \ u(4,0) = 0,$$

we have $h\ = 1,\ \ k = 1,\ \ and\, c = \frac{1}{2}$, using these values we can find

$\alpha = \frac{kc^2}{h^2} = \frac{1}{2}$ By Schmidt method α must be $\frac{1}{2}$

By Schmidt Formula the following Results are Obtained, $u_{i,j+1} = \frac{1}{2}\left[u_{i-1,j} + u_{i+1,j}\right]$

When $i = 1,2,3$ and $j = 0$,

$$u_1 = \frac{1}{2}[0 + 4] = 2$$

$$u_2 = \frac{1}{2}[3+3] = 3$$

$$u_3 = \frac{1}{2}[4+0] = 2$$

When $i = 1{,}2{,}3 \; and \; j = 1,$

$$u_5 = \frac{1}{2}[0+3] = 1.5$$

$$u_6 = \frac{1}{2}[2+2] = 2$$

$$u_7 = \frac{1}{2}[3+0] = 1.5$$

When $i = 1{,}2{,}3 \; and \; j = 2$

$$u_9 = \frac{1}{2}[0+2] = 1$$

$$u_{10} = \frac{1}{2}[1.5+1.5] = 1.5$$

$$u_{11} = \frac{1}{2}[2+0] = 1$$

Now Solve by Crank-Nicolson Formula

$$-\alpha u_{i-1,j+1} + (2+2\alpha)u_{i,j+1} - \alpha u_{i+1,j+1} = \alpha u_{i-1,j} + (2-2\alpha)u_{i,j} + \alpha u_{i+1,j}$$

Put $\alpha = \frac{1}{2}$

$$-u_{i-1,j+1} + 6\,u_{i,j+1} - u_{i+1,j+1} = u_{i-1,j} + 2\,u_{i,j} + u_{i+1,j}$$

When $i = 1{,}2{,}3 \; and \; j = 0,$

$$-0 + 6\,u_{1,1} - u_{2,1} = u_0 + 2\,u_1 + u_2 \quad (i)$$

$$-u_{1,1} + 6\,u_{2,1} - u_{3,1} = u_1 + 2\,u_2 + u_3 \quad (ii)$$

$$-u_{2,1} + 6\,u_{3,1} - u_{4,1} = u_2 + 2\,u_3 + u_4 \quad (iii)$$

When the initial conditions are given in $(i), (ii), (iii),$ we get.

$$6\,u_1 - u_2 + 0u_3 = 10 \qquad (i)$$

$$-u_1 + 6\,u_2 - u_3 = 14 \qquad (ii)$$

$$-u_2 + 6\,u_3 + 0\,u_4 = 10 \qquad (iii)$$

When we solve the equations $(i), (ii), (iii),$ by manual or by calculator it gives that,

$$u_1 = 2.176470588,\ u_2 = 3.058823529,\ u_3 = 2.176470588,$$

When $i = 1{,}2{,}3\ and\ j = 1,$

In the same way can we find,

$$u_4 = 1.615916955,\quad u_5 = 2.283737023,\ u_6 = 1.615916955,$$

RESULTS

Table 1: First Step Using the Analytical Solution and Crank-Nicolson, Bender-Schmidt

Analytical result	2.1356	3.032585	2.1356
Schmidt	2	3	2
Crank	2.176470588	3.0588233529	2.176470588

Table 2: Second Step Using the Crank-Nicolson and Bender-Schmidt

Schmidt	1.5	2	1.5
Crank	1.615916955	2.283737023	1.615916955
	u_1	u_2	u_3

CONCLUSIONS

On the basis of the above discussion we get the result obtained by analytical methods is always providing accurate solution but numerical solution always providing approximate result. But among these numerical methods Crank-Nicolsan method was providing fast convergence in comparison to Bender-Schdimit method. Since it is not possible to solve every partial differential equation analytically so numerical methods providing a good agreement in those cases where solutions not exist or we are unable to solve partial differential equation analytically.

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